Watson-Sommerfeld Transformation for Many-Particle Scattering Amplitudes*

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(Received 30 September 1963; revised manuscript received 24 January 1964)

The continuation of many-particle partial-wave scattering amplitudes to complex values of the total angular momentum is discussed in the framework of potential scattering. We show that if there is a continuation for which a Watson-Sommerfeld transformation of the full scattering amplitude can be made, then it is unique and determines the behavior of the amplitude for large values of any single scattering angle. The continuation of the partial-wave Schrödinger equation to complex values of the angular momentum is discussed, and the results are generalized to the case when exchange forces are present. As a simple application of the results, we discuss a crude nuclear model to illustrate how sequences of rotational levels can be described by Regge trajectories.

I. INTRODUCTION

A SIMPLE description of two-particle potential scattering amplitudes at large momentum transfers has been given by Regge and others in terms of poles in the partial-wave amplitudes at complex values of the angular momentum.[1–4] The question naturally arises as to whether this simplicity persists when multiparticle scattering processes are considered.

This question is of interest for several reasons. The large momentum-transfer behavior of a two-particle scattering amplitude is important in relativistic problems where it represents large energy of the cross channel. In problems at relativistic energies, however, two-particle channels are always coupled to channels of higher particle number by the possibility of particle production. It is important to answer the question of whether this coupling of multiparticle processes gives rise to cuts in the angular momentum plane and if so, to understand the dynamical mechanism which produces them.

Multiparticle potential scattering provides a simple starting point for investigating the analytic properties of many-particle amplitudes and a model in which some degree of rigor can presumably be obtained. If past experience is a guide, potential scattering should possess many of the features of the relativistic problem, but not all of them. A potential scattering model should serve to isolate the dynamical mechanism through which these relativistic features arise.

The continuation of multiparticle potential scattering amplitudes to complex values of the angular momentum is of interest in its own right because of possible application to scattering problems in which nuclei are involved. For instance, one would like to understand in more detail how sequences of nuclear rotational levels can be correlated with definite Regge trajectories. Before any useful approximate descriptions can be made, it is desirable to know the analytic properties of the exact amplitudes.

Some general properties of the continuation of many-particle scattering amplitudes in the total angular momentum by means of the Schrödinger equation are discussed in this paper. For simplicity, we have considered amplitudes which involve spinless and nonidentical particles.

In Sec. II the partial-wave Schrödinger equation is continued to complex values of the angular momentum. It is shown that if there is an analytic continuation of the partial-wave amplitude which determines the asymptotic behavior of the full amplitude for large values of one scattering angle, then it determines the asymptotic behavior in any other single scattering angle. In Sec. III the generalization needed to include exchange forces is given. Section IV contains an application of some of the results to a crude nuclear model illustrating how sequences of rotational levels with ground-state spins greater than zero or one-half can be described by Regge trajectories.

II. THE PARTIAL-WAVE EXPANSION OF A MANY-PARTICLE SCATTERING AMPLITUDE

Every many-particle system has three degrees of freedom which correspond to total rotations. Invariance of the Hamiltonian under these rotations implies the conservation of the total angular momentum L and its projection on a space-fixed axis M. The coordinates which specify rotations of the entire system we may take to be three Euler angles φ, θ, ψ, relating a "space-fixed" and a "body-fixed" set of Cartesian coordinates.[5] There are many ways of specifying these angles. Each

* This work supported in part by the U. S. Atomic Energy Commission.
† National Science Foundation Predoctoral Fellow.

[1] A. Bottino, A. Longoni, and T. Regge, Nuovo Cimento **23**, 954 (1962).

[2] J. Charap and E. Squires, Ann. Phys. (N. Y.) **20**, 145 (1962); **21**, 8 (1963).

[3] L. Favella and M. Reineri, Nuovo Cimento **23**, 616 (1962).

[4] A. Jaffe and Y. Kim, Phys. Rev. **127**, 2261 (1962).

[5] If L_x, L_y, L_z are components of the total angular momentum $\mathbf{L}$, a rotation R through φ, θ, ψ is given by

$$R(\varphi,\theta,\psi)=e^{i\varphi L_z}e^{i\theta L_y}e^{i\psi L_z}.$$

With appropriate relabeling, the conventions used in this paper are those of A. Edmonds, *Angular Momentum in Quantum Mechanics* (Princeton University Press, Princeton, New Jersey, 1957).

way corresponds to a definite convention as to how the body-fixed axes are fixed in the system of particles.[6]

A complete set of commuting observables conjugate to the three Euler angles are $\mathbf{L}^2$, L_z, and L_z', the total angular momentum and its projection on the space-fixed and body-fixed z axes, respectively. The eigenfunctions of these quantities are discussed in the Appendix and defined by

$$\begin{aligned} \mathbf{L}^2 D_{MK}{}^L(\varphi,\theta,\psi) &= L(L+1) D_{MK}{}^L(\varphi,\theta,\psi)\,, \\ L_z D_{MK}{}^L(\varphi,\theta,\psi) &= M D_{MK}{}^L(\varphi,\theta,\psi)\,, \\ L_z' D_{MK}{}^L(\varphi,\theta,\psi) &= K D_{MK}{}^L(\varphi,\theta,\psi)\,. \end{aligned} \tag{2.1}$$

Consider the scattering of N particles whose initial momenta $\mathbf{p}_1, \mathbf{p}_2, \cdots, \mathbf{p}_N$ will be denoted collectively by $\mathbf{p}$. The scattering wave function is a function of the coordinates $\mathbf{r}_1, \mathbf{r}_2, \cdots, \mathbf{r}_N$ denoted collectively by $\mathbf{r}$, and the momenta of the incoming wave; $\psi=\psi(\mathbf{r},\mathbf{p})$. Denoting the Euler angles of the coordinates by Ω_r, and those of the momenta by Ω_p, we may expand

$$\begin{aligned} \psi(\mathbf{r},\mathbf{p}) &= (8\pi^2)^{-1} \sum (2L+1) \psi_{K'K}{}^L(r,p) D_{MK'}{}^L(\Omega_r) \times D_{MK}{}^{L*}(\Omega_p) \\ &= (8\pi^2)^{-1} \sum (2L+1) \psi_{K'K}{}^L(r,p) D_{KK'}{}^L(\Omega_{rp})\,. \end{aligned} \tag{2.2}$$

The sum ranges over positive integral values of L and integer M, K, K' such that $|M|\leq L$, $|K|\leq L$, $|K'|\leq L$. The last line follows from the addition theorem[7] for the eigenfunctions of $\mathbf{L}^2$, L_z, L_z', where Ω_{rp} are the Euler angles of $\mathbf{r}$ with the body-fixed axes for $\mathbf{p}$ used as the space-fixed axes. Similarly, the scattering amplitude may be expanded as

$$\langle \mathbf{p}'|T|\mathbf{p}\rangle = (8\pi^2)^{-1} \sum (2L+1) T_{K'K}{}^L(p',p) \times D_{KK'}{}^L(\Omega_{p'p})\,. \tag{2.3}$$

If $\mathcal{T}$ is the kinetic energy, the Schrödinger equation is

$$[\mathcal{T}(\mathbf{r}) - E + V(r)]\psi(\mathbf{r},\mathbf{p}) = 0\,. \tag{2.4}$$

Since the kinetic energy is at most quadratic in the components of $\mathbf{L}$, it is straightforward to use Eqs. (A1) to project out the angles ψ and φ obtaining a partial-wave equation

$$\sum_{K''} [\mathcal{T}_{K'K''}{}^L(r) + (V(r)-E)\delta_{K'K''}] \times \psi_{K''K}{}^L(r,p) = 0\,. \tag{2.5}$$

The potential energy and energy terms are rotational scalars and hence diagonal in K. Again the sum is over $|K''|\leq L$.

For the case of two-particle potential scattering, the scattering amplitude for complex values of the angular momentum is found by writing a radial Schrödinger equation in which L appears as a parameter and then solving for the scattering solution at complex values of this parameter. The amplitude can be read off of the scattering solution and is the unique one for which a Watson-Sommerfeld transformation of the full amplitude can be made to determine its large momentum-transfer behavior.

We will try and follow this program for the case of many particles. A difficulty is that in Eq. (2.5), L not only appears as a parameter but also determines the number of coupled equations. Specifically, there are $(2L+1)^2$ equations, one for each value of K and K'. How can L become complex if it determines the number of equations?

The difficulty can be overcome by defining additional unphysical equations and unphysical wave functions which are not coupled to the physical quantities for integral values of L and such that the number of equations does not depend on L. Indeed, since Eq. (2.5) contains only the matrix elements from Eq. (A1), we can obtain a sensible set of equations at complex L simply by ignoring the restrictions on K, K', and K'' in Eq. (2.5) and allowing the sum to range over all integer values.

An equivalent way of doing this is to continue the $D_{KK'}{}^L$ to complex L so that they form a matrix with K and K' ranging over all integers from $-\infty$ to $+\infty$. This is discussed in the Appendix. Since $\mathbf{L}^2$ commutes with the Hamiltonian, we can demand that

$$\psi^L(\mathbf{r},\mathbf{p}) = (8\pi^2)^{-1} \sum_{KK'=-\infty}^{\infty} (2L+1) \psi_{K'K}{}^L(r,p) \times D_{KK'}{}^L(\Omega_{rp}) \tag{2.6}$$

solve the full Schrödinger equation (2.4). Projecting out the angles ψ and φ, we arrive at the infinite set of coupled equations mentioned above. Since the $D_{KK'}{}^L$ at integral values of L are nonzero only if K and K' are simultaneously greater than or less than L in absolute value, the equations with $|K|\leq L$, $|K'|\leq L$ will decouple from the rest and coincide with the physical ones. This can also be seen from Eq. (A1). The presence of unphysical wave functions is familiar from the problems involving the scattering of particles with spin.[8] L now occurs only as a parameter in the larger set of equations and the solutions can be examined at its complex values.

The asymptotic behavior of the full amplitude can be determined from the singularities in the angular momentum plane if the amplitude has the following properties:

(i) $T_{-K'K}{}^L D_{K'K}{}^L(\Omega)$ is an analytic function of L with singularities consisting of poles and cuts confined to the region $\mathrm{Re}L < L_0$ for some L_0.

(ii) $T_{K'K}{}^L$ decreases sufficiently fast as $|L|\to\infty$ so that the integral in the Watson-Sommerfeld trans-

[6] For some examples, see the following paper [J. B. Hartle, Phys. Rev. **134**, B620 (1964)] and G. Derrick and J. Blatt, Nucl. Phys. **8**, 310 (1958).

[7] See Ref. 5, p. 63.

[8] M. Gell-Mann, *Proceedings of the 1962 Annual International Conference on High-Energy Physics at CERN*, edited by J. Prentki (CERN, Geneva, 1962); see also Ref. 2.

formation over a large semicircle in the right-half plane tends to zero as its radius becomes large.

In the following paper we will outline a proof that by solving the Schrödinger equation continued to complex L as above, an analytic continuation of certain amplitudes can be obtained with properties (i) and (ii), and moreover, that the possibility of cuts can be dispensed with.

Under assumption (i), the partial-wave expansion can be written

$$\langle \mathbf{p}'|T|\mathbf{p}\rangle=\frac{1}{16\pi^2 i}\sum_{KK'=-\infty}^{\infty}\int_{C_{KK'}}\frac{(2L+1)dL}{\sin\pi L}\times T_{-K'K}{}^{L}D_{K'K}{}^{L}(\omega).$$

Here, we have used the symmetry relations of the $D_{KK'}{}^{L}(\Omega)$ for integer L,[9] so that if $\Omega_{p'p}$ denotes the angles φ, θ, ψ; ω denotes $\pi-\psi$, $\pi-\theta$, φ. The contour $C_{KK'}$ encloses the real axis $\mathrm{Re}L>[\max(|K|,|K'|)-\frac{1}{2}]$. It is assumed to exclude all poles but those of $\sin\pi L$. If a cut crosses the real axis in this region, it would be necessary to subtract out its contribution.

Under assumption (ii), the contour can be deformed to a line Γ parallel to the imaginary axis at $\mathrm{Re}L=-\frac{1}{2}$, yielding

$$\begin{aligned}\langle \mathbf{p}'|T|\mathbf{p}\rangle=&\frac{1}{16\pi^2 i}\int_{\Gamma}\frac{(2L+1)dL}{\sin\pi L}\sum_{K'K}T_{-K'K}{}^{L}\\&\times D_{K'K}{}^{L}(\omega)+\sum_{KK'}\sum_{k}(16\pi^3 i)^{-1}(-1)^k(2k+1)\\&\times T_{-K'K}{}^{k}D_{K'K}{}^{k}(\omega)+\sum_i\int d\tau[\sin\pi\alpha_i(\tau)]^{-1}\\&\times\sum_{K'K}\rho_{K'K}{}^{i}(\tau)D_{K'K}{}^{\alpha_i(\tau)}(\omega)+\sum_n(\sin\pi\alpha_n)^{-1}\\&\times\sum_{K'K}B_{K'K}{}^{\alpha_n}D_{K'K}{}^{\alpha_n}(\omega),\end{aligned}\tag{2.7}$$

where $\rho_{K'K}{}^{i}$ and $B_{K'K}{}^{\alpha_n}$ are simply related to the discontinuities and residues of $T_{-K'K}{}^{L}$, respectively. The sum over k ranges over integers such that $0\leq k<\max(|K'|,|K|)$.

For large values of $z=\cos\theta$, and integer k such that $|k|\leq\max(|K'|,|K|)$, $D_{K'K}{}^{L}$ behaves like (see Appendix)

$$D_{K'K}{}^{k}(\omega)\sim g_{K'K}{}^{k}(\varphi,\psi)z^{-k-1}\tag{2.8a}$$

and like

$$D_{K'K}{}^{L}(\omega)\sim g_{K'K}{}^{L}(\varphi,\psi)z^{L}\tag{2.8b}$$

for other L. The asymptotic behavior of the amplitude is then determined by the position of the singularity furthest right in the L plane. For instance, suppose it is a pole at position α, then

$$\langle \mathbf{p}'|T|\mathbf{p}\rangle\sim\left(\sum_{K'K}B_{K'K}{}^{\alpha}g_{K'K}{}^{\alpha}\right)z^{\alpha}.\tag{2.9}$$

The continuation which satisfies conditions (i) and (ii) and thus determines the asymptotic behavior is unique. The uniqueness may be established by considering the partial-wave expansion of

$$\int d\psi\int d\varphi e^{-iM\varphi}e^{-iN\psi}\langle \mathbf{p}'|T|\mathbf{p}\rangle\tag{2.10}$$

and applying the discussion of Squires.[10] Briefly, suppose there are two continuations $T_{MN}{}^{L}(1)$ and $T_{MN}{}^{L}(2)$ which satisfy the conditions (i) and (ii) and agree with the physical values on the integers; then

$$T_{MN}{}^{L}(1)+(L-L_1)^{-1}[T_{MN}{}^{L}(1)-T_{MN}{}^{L}(2)]\tag{2.11}$$

will also. The quantity in Eq. (2.10) has a unique asymptotic behavior which we may say is weaker than z^{L_0}. If we take $\mathrm{Re}L_1>\mathrm{Re}L_0$, we must have

$$T_{MN}{}^{L_1}(1)=T_{MN}{}^{L_1}(2)\tag{2.12}$$

in order for the continuation (2.11) to give the correct asymptotic behavior. Since this holds for any L_1 for which $\mathrm{Re}L_1>\mathrm{Re}L_0$, it must hold everywhere.

In a two-particle scattering problem, there is essentially only one scattering angle. For many-particle scattering, however, there are many. Each may be characterized as an angle between z axes fixed in the initial and final systems of particles. As we have mentioned before, there are many ways of choosing how the body-fixed axes are fixed in the system of particles and hence many scattering angles. We will now show that if a continuation exists which determines the asymptotic behavior in one scattering angle, then it determines the behavior in all.

From a single choice of body-fixed axes, all others can be found by rotations in the body-fixed frames of the coordinates and momenta. Let Ω_1 and Ω_2 be two choices of Euler angles and W the rotation which sends Ω_1 into Ω_2. For integer values of L, one knows how the $D_{MK}{}^{L}$ transform under this rotation since they form a representation of the rotation group

$$\begin{aligned}D_{MK}{}^{L}(\Omega_2)&=D_{MK}{}^{L}(W\Omega_1)\\&=\sum_N D_{MN}{}^{L}(W)D_{NK}{}^{L}(\Omega_1),\end{aligned}\tag{2.13}$$

the sum ranging from $N=-L$ to $N=+L$. Several authors[16] have shown that the $D_{MK}{}^{L}$ for complex L also satisfy Eq. (2.13) for a certain range of angles, the sum then being extended from $N=-\infty$ to $N=+\infty$. Another derivation is given in the Appendix. For angles outside this range a similar decomposition of $D_{MK}{}^{L}(W\Omega_1)$ can be made [see Eq. (A12)] and the relations derived below hold with slight alteration.

To determine how the partial-wave functions $\psi_{K'K}{}^{L}$ transform under a new choice of body-fixed axes, consider the rotational scalar $\psi^{L}(\mathbf{r},\mathbf{p})$ of Eq. (2.6). Denote by W_1 the rotation in the body-fixed coordinate frame and by W_2 the rotation in the body-fixed momentum

[9] See Ref. 5, p. 60.

[10] E. Squires, Nuovo Cimento 24, 242 (1962).

frame. These will be functions of the internal variables r and p, respectively. Setting $\Omega_{rp}=W_2\Omega_{rp}'W_1$, we have

$$\begin{aligned}\psi^L(\mathbf{r},\mathbf{p})&=(8\pi^2)^{-1}\sum_{K'K}(2L+1)\psi_{K'K}{}^L(r,p)\times D_{KK'}{}^L(W_2\Omega_{rp}'W_1)\\&=(8\pi^2)^{-1}\sum_{K'K}(2L+1)\psi_{K'K}{}^L(r,p)\times\sum_{N'N}D_{KN}{}^L(W_2)D_{NN'}{}^L(\Omega_{rp}')\times D_{N'K'}{}^L(W_1).\end{aligned}\quad(2.14)$$

The wave function therefore transforms like

$$\psi'_{N'N}{}^L(r,p)=\sum_{K'K}D_{N'K'}{}^L(W_1)\psi_{K'K}{}^L(r,p)\times D_{KN}{}^L(W_2).\quad(2.15)$$

Similarly, if rotations are performed in the initial and final momenta so that $\omega=Y_2\omega'Y_1$, the amplitude transforms like

$$T'_{-N'N}{}^L=\sum_{K'K}D_{K'N'}{}^L(Y_2)T_{-K'K}{}^LD_{NK}{}^L(Y_1).\quad(2.16)$$

Since terms with different complex angular momenta L do not mix under rotations, a singularity which determines the asymptotic behavior in one representation will also determine it in any other. For instance, if the amplitude has a large z behavior given by Eq. (2.9), then the asymptotic behavior in any other scattering angle z' would be given by

$$\langle\mathbf{p}'|T|\mathbf{p}\rangle\sim\left(\sum_{K'K}B'_{K'K}{}^\alpha g_{K'K}{}^\alpha\right)z'^\alpha,\quad(2.17)$$

where

$$B'_{K'K}{}^\alpha=\sum_{N'N}D_{N'K'}{}^\alpha(Y_2)B_{N'N}{}^\alpha D_{KN}{}^\alpha(Y_1)\quad(2.18)$$

and Y_1 and Y_2 are the rotations appropriate to the changes of body-fixed axes in the initial and final states, respectively.

Since the continuation which determines the asymptotic behavior in a given scattering angle is unique, we conclude that there is one unique continuation (if it exists) which determines the asymptotic behavior in any scattering angle. This conclusion does not depend on the nonrelativistic model being considered here. It is a general property of the Watson-Sommerfeld transformation and will hold for the relativistic amplitudes if they satisfy the assumptions (i) and (ii).

III. EXCHANGE FORCES

In the nonrelativistic limit of a problem in which particles can be created and destroyed at a vertex, certain nonlocal potentials can be expected to occur. In particular, there is the class of potentials which rearrange the positions of the particles but otherwise act in a local way. Such potentials have the form

$$V=\sum_n\lambda V^n(\mathbf{r}_1,\cdots,\mathbf{r}_N)P_n,\quad(3.1)$$

where P_n is a member of the permutation group on N objects. We will now consider the simple generalization of the preceding section necessary to continue to complex values of the angular momentum amplitudes produced by such forces.

The full Schrödinger equation (2.4) can be written

$$[\mathcal{T}(\mathbf{r})-E+\lambda\sum_n V^n(\mathbf{r})P_n]\psi(\mathbf{r},\mathbf{p})=0.\quad(3.2)$$

Operate on this equation with the permutation P_i. Since the permutations form a group, we have

$$P_iP_n=\sum_j\mathcal{P}_{ij}{}^nP_j,\quad(3.3)$$

where $\mathcal{P}_{ij}{}^n$ is a representation of P_n. The indicated operation then gives:

$$\sum_j[(\mathcal{T}_i-E)\delta_{ij}+\lambda\sum_n V_i{}^n(\mathbf{r})\mathcal{P}_{ij}{}^n]\psi_j(\mathbf{r},\mathbf{p})=0,\quad(3.4)$$

where

$$\mathcal{T}_i(\mathbf{r})=\mathcal{T}(P_i\mathbf{r}),\quad V_i{}^n(\mathbf{r})=V^n(P_i\mathbf{r}),\quad \psi_i(\mathbf{r},\mathbf{p})=\psi(P_i\mathbf{r},\mathbf{p}).$$

This is a set of six coupled equations on the six unknowns $\psi_i(\mathbf{r})$ involving only local potentials. The corresponding set of partial-wave equations can then be continued to complex L as discussed in Sec. II, and every coefficient will be bounded by a polynomial for large $|L|$. The presence of exchange operators, on the other hand, could entail factors with asymptotic behaviors like $e^{\pm i\pi L}$ and perhaps lead to a violation of assumption (ii). The original Eq. (3.2) was solved with the boundary condition that it approach a plane-wave φ as the potential tends to zero:

$$\psi(\mathbf{r},\mathbf{p})\to\varphi(\mathbf{r},\mathbf{p}),\quad\lambda\to0.\quad(3.5)$$

The same solution may be generated by solving Eq. (3.4) with the boundary conditions

$$\psi_i(\mathbf{r},\mathbf{p})\to\varphi_i(\mathbf{r},\mathbf{p})=\varphi(P_i\mathbf{r},\mathbf{p}),\quad\lambda\to0.\quad(3.6)$$

If $P_1=1$, the scattering solution is then given by $\psi(\mathbf{r},\mathbf{p})=\psi_1(\mathbf{r},\mathbf{p})$. Equivalently, we can write

$$\psi(\mathbf{r},\mathbf{p})=\sum_j\psi_{1j}(\mathbf{r},\mathbf{p}),\quad(3.7)$$

where ψ_{ij} has the boundary condition

$$\psi_{ij}(\mathbf{r},\mathbf{p})\to\delta_{ij}\varphi_i(\mathbf{r},\mathbf{p}),\quad\lambda\to0.\quad(3.8)$$

For a plane wave, a permutation acting on the coordinates is the same as the conjugate permutation acting on the momenta

$$\varphi_i(\mathbf{r},\mathbf{p})=\varphi(P_i\mathbf{r},\mathbf{p})=\varphi(\mathbf{r},P_i{}^\dagger\mathbf{p}).\quad(3.9)$$

Every permutation of the momenta can be written as the product of a transformation Q_i which changes the lengths of the relative momentum vectors p and a rotation of the body-fixed axes R_i.

$$P_i=Q_iR_i.\quad(3.10)$$

The partial-wave expansion of $\varphi_i(\mathbf{r},\mathbf{p})$ can then be written

$$\varphi_i(\mathbf{r},\mathbf{p})=(8\pi^2)^{-1}\sum_{LK'K}(2L+1)\varphi_{K'K}{}^L(r,Q_i{}^\dagger p)\times D_{KK'}{}^L(R_i{}^\dagger\Omega_{rp}).\quad(3.11)$$

The boundary condition for the partial-wave functions of $\psi_{ij}(\mathbf{r},\mathbf{p})$ are then

$$\psi_{iK',jK}{}^L(r,p) \to \delta_{ij} \sum_{K''} \varphi_{K'K''}{}^L(r,Q_j{}^\dagger p) \times D_{K''K}{}^L(R_j{}^\dagger). \quad (3.12)$$

Consider the wave functions $\tilde{\psi}_{iK',jK}{}^L$, defined by the boundary conditions

$$\tilde{\psi}_{iK',jK}{}^L \to \delta_{ij}\varphi_{K'K}{}^L(r,Q_j{}^\dagger p). \quad (3.13)$$

Since the Schrödinger equation is linear, multiplication of the boundary condition by the constant matrix $D_{K'K}{}^L(R_j{}^\dagger)$ only multiplies the solution by the same matrix.

$$\psi_{iK',jK}{}^L = \sum_{K''} \tilde{\psi}_{iK',jK''}{}^L D_{K''K}{}^L(R_j{}^\dagger). \quad (3.14)$$

The scattering solution, Eq. (3.7), can then be written

$$\psi(\mathbf{r},\mathbf{p}) = (8\pi^2)^{-1} \sum_{LK'Kj} (2L+1)\tilde{\psi}_{1K',jK}{}^L \times D_{KK'}{}^L(R_j{}^\dagger\Omega_{rp}). \quad (3.15)$$

Similarly, the partial-wave expansion of the amplitude becomes

$$\langle \mathbf{p}'|T|\mathbf{p}\rangle = (8\pi^2)^{-1} \sum_{LK'Kj} (2L+1)\tilde{T}_{1K',jK}{}^L(p',p) \times D_{KK'}{}^L(R_j{}^\dagger\Omega_{p'p}), \quad (3.16)$$

where the amplitudes $\tilde{T}_{iK',jK}{}^L$ are computed from the scattering solutions $\tilde{\psi}_{iK',jK}{}^L$.

Equation (3.4) has only local potentials. The boundary condition (3.13) has the same large $|L|$ behavior as that for a partial-wave expansion of (3.5). If there is an analytic continuation of the problem of a single three-particle channel with local potentials which satisfies criteria (i) and (ii) of Sec. II, it is not unreasonable to expect that the same result holds for the six-coupled three-particle channels of Eq. (3.4). $\tilde{T}_{1K',jK}{}^L$ can therefore be assumed to satisfy conditions (i) and (ii) and the Watson-Sommerfeld transformation of Eq. (3.16) can be performed.

For nonidentical particles, Eq. (3.4) may not be further decoupled and a given pole will appear in all the analytically continued partial-wave amplitudes. However, if the particles have equal masses and

$$[P_i,V]=0, \quad (3.17)$$

as in the case of identical particles, a further decomposition of the set of equations may be obtained by reducing the representation $\mathcal{P}_{ij}{}^n$. The amplitude can be decomposed into a sum of terms [linear combinations of the arguments of (3.16)], one for each irreducible representation of the permutation group. The poles in one of these terms need not appear in any other. This is the analog of signature in the two-particle case. If the particles are truly identical (and not, for example, differently charged and hence distinguishable pions interacting by nuclear forces), only the poles of the symmetric or antisymmetric amplitudes would correspond to physical states.

Let us consider the sample example of three particles using the choice of Euler angles discussed by Blatt and Derrick.[6] In these coordinates, the body-fixed z axis is taken normal to the triangle formed by the three particles and directed so that a right-handed screw will advance along it if turned successively through particles 1, 2, 3, and back to 1. Orthogonal x and y axes are defined invariantly as discussed by these authors, for instance, by taking the x axis to lie along the principle axis of the triangle with greatest moment of inertia.

With these definitions, an interchange of particles changes only the sign of the z axis.

$$\begin{aligned} P_i D_{K'K}{}^L(\pi-\psi,\pi-\theta,\varphi) &= D_{K'K}{}^L(\pi-\psi,\pi-\theta,\varphi), \quad \delta_i=+1 \\ &= D_{K'K}{}^L(\pi-\psi,\theta,\varphi), \quad \delta_i=-1, \quad (3.18)\end{aligned}$$

where δ_i is $+1$ or -1 as the number of interchanges in P_i is even or odd. The remaining variables p may be taken to be the relative momenta p_{12}, p_{23}, p_{13}. A Regge pole term in the full amplitude then has the form

$$(\sin\pi\alpha)^{-1} \sum_{K'K} [F_{K'K}{}^\alpha(p',p)D_{K'K}{}^\alpha(\pi-\psi,\pi-\theta,\varphi) + G_{K'K}{}^\alpha(p',p)D_{K'K}{}^\alpha(\pi-\psi,\theta,\varphi)]. \quad (3.19)$$

If conditions (3.17) are satisfied, there will be three types of poles for the symmetric, antisymmetric and mixed representations of the permutation group on three objects. Equation (3.19) will then have the corresponding symmetry.

IV. A SIMPLE NUCLEAR MODEL

In this section we discuss a simple consequence of the decoupling of the physical and unphysical solutions to Schrödinger's equation at integral values of L. On the basis of a crude model, nuclear rotational levels are correlated with definite Regge trajectories. The presence of the unphysical solutions at integral values of L enables the theory to incorporate sequences of rotational levels whose lowest state possesses a spin greater than zero.

Let us first examine the implications of the unitarity condition for the behavior of a trajectory as it passes through an integer. The unitarity condition can be continued to complex L by an application of Carlson's theorem[11] and assumption (ii) of Sec. II. One of the consequences of the resulting unitarity relation is that the residues of a Regge pole in a many-channel amplitude factor.[12]

$$B_{K'K}{}^\alpha(p',p) = b_{K'}{}^\alpha(p')b_K{}^\alpha(p). \quad (4.1)$$

Suppose a trajectory $\alpha(E)$ passes through a positive integer α_0 at energy E_0. Since there is no coupling between the physical and unphysical solutions at an

[11] R. Boas, *Entire Functions* (Academic Press, Inc., New York, 1954), p. 153.

[12] V. Gribov and I. Ya. Pomeranchuk, Phys. Rev. Letters **8**, 343 (1962).

integer, the amplitudes which connect these states must vanish.

$$T_{KK'}{}^L = T_{K'K}{}^L = 0 \quad |K| \leq L, \quad |K'| > L, \quad L \text{ integer}. \quad (4.2)$$

If a pole appears in one physical amplitude, it must appear in all of them, and similarly for the unphysical amplitudes. Factorization then implies that either $b_K{}^{\alpha_0}=0$ for all K such that $|K|>\alpha_0$ or $b_K{}^{\alpha_0}=0$ for all K such that $|K|\leq\alpha_0$. In the first case, the pole appears in the physical and not the unphysical amplitudes. In the second case, the opposite is true. Every Regge trajectory as it passes through an integer thus has a choice. It can either appear in the physical or unphysical amplitudes but not both.

At integral values of L, it is only the poles in the physical amplitudes which correspond to bound states. Thus, as a trajectory crosses and integer value, it does not always correspond to a physical state. Which choice it makes at a given integer depends on the particular dynamics at hand.

To illustrate the application of these alternative possibilities for a trajectory at the integers, we will consider a standard model from nuclear physics. The model consists of a single-particle scattering off a rigid rotator core.[13,14] For simplicity, we will take the particle outside of the core to be spinless. The wave function is then a function of the three Euler angles which specify the orientation of the core, φ, θ, ψ; and the position of the particle, $\mathbf{r}$. If this position is specified in the body-fixed system, then the three Euler angles are conjugate to the total angular momentum $\mathbf{L}$. The partial-wave expansion is again

$$\psi(\mathbf{r},\varphi,\theta,\psi) = \sum_{LMK} \psi_K{}^L(\mathbf{r}) D_{MK}{}^L(\varphi,\theta,\psi). \quad (4.3)$$

If $\mathbf{I}$ is the angular momentum of the rotator, J_a, $a=1$, 2, 3, its moments of inertia, and $\mathbf{l}$ the angular momentum of the particle, the Hamiltonian for the system can be written

$$H = H_p + H_{\text{rot}},$$
$$H_p = -\frac{1}{2m}\left(\frac{d^2}{dr^2}+\frac{2}{r}\frac{d}{dr}-\frac{\mathbf{l}^2}{r^2}\right)+V(\mathbf{r}), \quad (4.4)$$
$$H_{\text{rot}} = \sum_a I_a{}^2/2J_a.$$

In the rotational part of the Hamiltonian, we can replace $\mathbf{I}$ by $\mathbf{L}-\mathbf{l}$. We first make the standard approximation that the nucleus is spheriodal

$$J_1 = J_2 = J, \quad (4.5)$$

and that the potential is symmetric under rotations about the body-fixed z axis and reflections in the body-fixed x-y plane. It is conventional to demand that wave functions must be invariant under rotations of the rotator about its figure axis, or equivalently that the figure axis component of the rotator angular momentum (L_3-l_3) is zero.

We can then write

$$H_{\text{rot}} = (1/2J)(\mathbf{L}-\mathbf{l})^2 = (1/2J)(\mathbf{L}^2+\mathbf{l}^2-2L_3l_3) + H_{\text{coup}}, \quad (4.6)$$
$$H_{\text{coup}} = J^{-1}(L_+l_- + L_-l_+).$$

In the first approximation, we neglect the coupling term so that L_3 is a good quantum number. The partial-wave Schrödinger equation is then

$$\left[\frac{1}{2m}\left(\frac{d^2}{dr^2}+\frac{2}{r}\frac{d}{dr}\right)-\left(\frac{1}{2mr^2}+\frac{1}{2J}\right)\mathbf{l}^2 - V(\mathbf{r}) + \mathcal{E}\right] \times \psi_K{}^L(\mathbf{r}) = 0, \quad (4.7)$$

with

$$\mathcal{E} = E - \frac{L(L+1)-2K^2}{2J}. \quad (4.8)$$

We wish to continue the amplitude to complex L. Clearly, this amounts to continuing the solutions of Eq. (4.7) to complex values of $\mathcal{E}$ with scattering boundary conditions. It is not important to consider the resulting analytic properties in detail. The model is an approximation to a many-particle system only for energies E near the bound-state energies; that is, only for complex L near those values for which bound states exist at the given E. The bound states of Eq. (4.7) are reflected as poles in the scattering amplitude in the quantity $\mathcal{E}$. Let $\mathcal{E}_K{}^0$ be the position of one such pole which will, in general, depend on K if the potential is only axially symmetric. To each such pole, we have a trajectory in the right-half plane.

$$L_K(E)+\tfrac{1}{2} = +\left[\tfrac{1}{4}+2K^2+2J(E-\mathcal{E}_K{}^0)\right]^{1/2}. \quad (4.9)$$

The levels which lie on a single trajectory are all members of a given rotational band.

Since K is a good quantum number for this approximate problem, the amplitude is diagonal in it. Each diagonal element $T_{KK}{}^L$ has its own trajectory $L_K(E)$. At the integers, the physical amplitudes are those for which $|K|\leq L$ and thus as $L_K(E)$ crosses an integer, it represents a physical bound state only for $L_K(E) \geq |K|$. Each trajectory thus represents a sequence of rotational levels with spin values $L=|K|$, $|K|+1$, $|K|+2$, $\cdots$. The trajectory itself, however, crosses all integer values.

A requirement usually imposed on this model is that the wave function be invariant under reflections in the x-y plane of the body-fixed system. The results of this assumption are in good agreement with experiment where the model applies. For $K\neq0$, the character of the spectrum remains the same; there is a sequence of

[13] A. Bohr and B. Mottelson, Kgl. Danske Videnskab. Selskab, Mat. Fys. Medd. 27, No. 16 (1953).

[14] This model has also been considered for the case of complex angular momentum and separable potentials by E. Kazes, Nuovo Cimento 27, 995 (1963).

levels with spins $L=|K|$, $|K|+1$, $|K|+2$, $\cdots$. For $K=0$, however, a new feature arises.

Let R denote a rotation about the body-fixed y axis by π. For $K=0$, $\psi_0{}^L(\mathbf{r})$ is invariant under rotations about the body-fixed z axis, so that

$$R\psi_0{}^L(\mathbf{r})=\nu\psi_0{}^L(\mathbf{r}), \tag{4.10}$$

where ν is the parity of $\psi_0{}^L(\mathbf{r})$. Because of the axial symmetry of the problem, the requirement of invariance under reflections is equivalent to a requirement of invariance under R. Taking $M=0$ for simplicity, the wave function satisfying this requirement is

$$\psi(\mathbf{r},\varphi,\theta,\psi)=\psi_0{}^L(\mathbf{r})[P_L(\cos\theta)+\nu P_L(-\cos\theta)]. \tag{4.11}$$

A given Regge trajectory in $\psi_0{}^L(\mathbf{r})$ will now represent a physical state only at every other integral value of L. It thus describes a sequence of levels with $L=0^+, 2^+, 4^+, \cdots$ or $L=1^-, 3^-, 5^-, \cdots$.

Suppose K is no longer a good quantum number. This happens when we consider the effect of H_{coup} or introduce small asymmetries in the moments of inertia J_1, J_2. The trajectories can no longer be computed exactly, but we can assume that for small symmetry breaking terms they do not depart strongly from Eq. (4.9). Each trajectory is characterized by a number $|K|$ which determines the diagonal amplitude $T_{KK}{}^L$ in which it appears as the symmetry breaking coupling disappears. As the coupling is turned on, the pole will appear in the amplitudes to which $T_{KK}{}^L$ is coupled. In general, this will be all the amplitudes except at the integers where the pole will appear in the physical amplitudes if K is physical ($|K|\leq L$) and in the unphysical amplitudes if K is unphysical ($|K|>L$).

Thus, even in the case when L_3 is not conserved, we have a sequence of rotational levels or bands correlated with a definite trajectory and characterized by a number K which is the spin of the lowest physical state in the band. For integer spins greater than K, the trajectory appears in the physical amplitudes, and in the unphysical amplitudes for spin values less than K.

ACKNOWLEDGMENTS

The author is grateful for several helpful conversations with J. Bardeen, Dr. J. Cornwall, Dr. A. Erdélyi, Dr. S. Frautschi, Dr. M. Gell-Mann, Dr. D. Sharp, and Dr. V. Singh.

APPENDIX: PROPERTIES OF THE ANGULAR MOMENTUM EIGENFUNCTIONS

Denote by φ, θ, ψ the three Euler angles conjugate to the total angular momentum $\mathbf{L}(\varphi,\theta,\psi)$. $\mathbf{L}$ is a differential operator on functions of these angles whose Cartesian components we denote by $L_i'(\varphi,\theta,\psi)$ in the body-fixed system and by $L_i(\varphi,\theta,\psi)$ in the space-fixed system.[15] The $D_{MK}{}^L(\varphi,\theta,\psi)$ are then defined for complex L by

$$D_{00}{}^L(\varphi,\theta,\psi)=P_L(\cos\theta),$$

$$\pm[\tfrac{1}{2}(L\pm M)(L\mp M+1)]^{1/2}D_{M\mp1,K}{}^L=L_{\mp}(\varphi,\theta,\psi)D_{MK}{}^L, \tag{A1}$$

$$\mp[\tfrac{1}{2}(L\pm K)(L\mp K+1)]^{1/2}D_{M,K\mp1}{}^L=L_{\pm}'(\varphi,\theta,\psi)D_{MK}{}^L,$$

where $L_{\pm}$, L_0 are the spherical components of $\mathbf{L}$. One choice of the branch of the square root must be made, say by taking it real for large real L. Since $\mathbf{L}^2$, $L_{\pm}$, L_0 satisfy the usual commutation relations, the $D_{MK}{}^L$ defined in this way also satisfy the differential equations (2.1). The solutions of these equations can be expressed in terms of hypergeometric functions. It can be checked from the definition (A1) that the $D_{MK}{}^L$ can be written

$$D_{MK}{}^L(\varphi,\theta,\psi)=e^{iM\varphi}d_{MK}{}^L(\cos\theta)e^{iK\psi},$$

$$d_{MK}{}^L(z)=\left[\frac{(L+M)!(L-K)!}{(L+K)!(L-M)!}\right]^{1/2}\times\left(\frac{1+z}{2}\right)^{(M+K)/2}\left(\frac{1-z}{2}\right)^{(M-K)/2}\times\frac{1}{(M-K)!}F\left(-L+M,\,L+M+1,\,M-K+1,\frac{1-z}{2}\right), \tag{A2}$$

for $M-K\geq0$, other cases being obtained from the symmetry relations for the $d_{MK}{}^L(z)$.[9] In particular, we have

$$D_{M0}{}^L(\varphi,\theta,\psi)=P_L{}^M(\cos\theta)e^{iM\varphi}\times[(L-M)!/(L+M)!]^{1/2}, \tag{A3}$$

$$D_{0K}{}^L(\varphi,\theta,\psi)=(-1)^KD_{K0}{}^L(\psi,\theta,\varphi)\quad M,K>0.$$

The $D_{MK}{}^L$ thus defined reduce to the usual values for integral L and $|M|\leq L$, $|K|\leq L$.

To prove that the $D_{MK}{}^L$ as defined for complex L by (A1) satisfy Eq. (2.13),[16] we begin with the addition theorem for Legendre functions[17]

$$P_L(x)=P_L(x_2)P_L(x_1)+2\sum_{N=1}^{\infty}\frac{(L-N)!}{(L+N)!}P_L{}^N(x_2)\times P_L{}^N(x_1)\cos N(\varphi_1+\psi_2-\pi). \tag{A4}$$

Here, $x=\cos\theta$, $x_1=\cos\theta_1$, $x_2=\cos\theta_2$, where φ, θ, ψ are the angles of the rotation resulting from successive rotations through $\varphi_1\theta_1\psi_1$ and $\varphi_2\theta_2\psi_2$. Applying (A3),

[15] For explicit expressions, see Ref. 5.

[16] Compare, J. Gunson (to be published), and E. Beltrametti and G. Luzzato, Nuovo Cimento **29**, 1003 (1963).

[17] A. Erdelyi *et al.*, *Higher Transcendental Functions* (McGraw-Hill Book Company, Inc., New York, 1953), p. 168.

this can be written

$$D_{00}^L(\varphi,\theta,\psi)=\sum_{N=-\infty}^{\infty} D_{0N}^L(\varphi_2,\theta_2,\psi_2)D_{N0}^L(\varphi_1,\theta_1,\psi_1). \quad \text{(A5)}$$

Consider the operator $L_+(\varphi_1,\theta_1,\psi_1)$ on functions φ, θ, ψ with φ_2, θ_2, ψ_2 remaining constant. $L_+(\varphi_1,\theta_1,\psi_1)$ may be expressed as a function of the new variables φ, θ, ψ as $L_+(\varphi_1(\varphi,\theta,\psi),\theta_1(\varphi,\theta,\psi),\psi_1(\varphi,\theta,\psi))$. To do this, regard the transformation of variables as a change of coordinates under which $\mathbf{L}$ transforms like a vector

$$L_i(\varphi,\theta,\psi)=\sum_j D_{ij}^{(1)}(\psi_2,\theta_2,\varphi_2) \times L_j(\varphi_1(\psi,\theta,\varphi),\theta_1(\psi,\theta,\varphi),\psi_1(\psi,\theta,\varphi)). \quad \text{(A6)}$$

The left-hand side of (A6) gives the components of $\mathbf{L}$ along the rotated axes while we need them along the old axes as functions of the new variables. To find these, multiply the new components by the matrix $[D^{(1)}(\psi_2,\theta_2,\varphi_2)]^{-1}$, thus obtaining

$$L_+(\varphi,\theta,\psi)=L_+(\varphi_1(\psi,\theta,\varphi),\theta_1(\psi,\theta,\varphi),\psi_1(\psi,\theta,\varphi)). \quad \text{(A7)}$$

Thus, applying $(L_+)^M$ to both sides of (A6) and multiplying both sides by a common normalization factor, we find

$$D_{M0}^L(\varphi,\theta,\psi)=\sum_{N=-\infty}^{\infty} D_{MN}^L(\varphi_2,\theta_2,\psi_2) \times D_{N0}^L(\varphi_1,\theta_1,\psi_1). \quad \text{(A8)}$$

A similar procedure with L_-, L_+', L_-' gives the desired result (2.13). The differentiation under the summation is justified since (A5) converges like $(\tan\theta_1/2\ \tan\theta_2/2)^N$ and hence uniformly in θ_1, θ_2 provided

$$0\leq\theta_1+\theta_2<\pi,\quad 0\leq\theta_1<\pi,\quad 0\leq\theta_2<\pi. \quad \text{(A9)}$$

A similar formula valid when

$$\pi<\theta_1+\theta_2,\quad 0\leq\theta_1<\pi,\quad 0\leq\theta_2<\pi \quad \text{(A10)}$$

can be derived from Eq. (A4) by replacing θ_1 by $\pi-\theta_1$, θ_2 by $\pi-\theta_2$ and noting that[15]

$$L_\pm(\varphi,\theta,\psi)=L_\mp(-\varphi,\pi-\theta,\psi), \quad L'_\pm(\varphi,\theta,\psi)=L'_\mp(\varphi,\pi-\theta,-\psi). \quad \text{(A11)}$$

One finds

$$D_{MK}^L(\varphi,\theta,\psi)=\sum_{N=-\infty}^{\infty} D_{-MN}^L(-\varphi_2,\pi-\theta_2,\psi_2) \times D_{N,-K}^L(\varphi_1,\pi-\theta_1,-\psi_1). \quad \text{(A12)}$$

More comprehensive discussions of these formulas can be found in Ref. 16.

For integer L, the elements of D_{MK}^L are nonzero only if either

$$|M|>L \quad and \quad |K|>L$$

or

$$|M|\leq L \quad and \quad |K|\leq L.$$

The second class of elements corresponds to the usual D_{MK}^L. At integers, the matrix D_{MK}^L is thus in block form.

The behavior of $d_{MK}^L(z)$ for large values of L and z has been discussed by Charap and Squires.[2] In order to display the behavior of Sec. II, we write d_{MN}^L as

$$d_{MK}^L(z)=\tfrac{1}{2}[f_{MK}^L(z)+f_{MK}^{-L-1}(z)],$$

where for $M-K\geq 0$

$$f_{MK}^L=[(L+M)!(L-K)!(L+K)!(L-M)!]^{-1/2} \times\left(\frac{z+1}{2}\right)^{(M+K)/2}\left(\frac{z-1}{2}\right)^{L-[(M+K)/2]}(-1)^{(M-K)/2} \times(2L+1)!F(-L+M,-L+K,-2L,2/(1-z)).$$

The desired asymptotic behavior then follows from the hypergeometric series.